

# Remarks on the paper "Non-existence of Shilnikov chaos in continuous-time systems"

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## Abstract

The present note refers to a result proposed in [2], and shows that the Theorem therein is not correct. We explain that a proof of that Theorem cannot be given, as the statement is not correct, and we underline a mistake occurring in their proof.

Since this note is supplementary to [2], the reader should consult this paper for further explanations of the matter and the symbols used.

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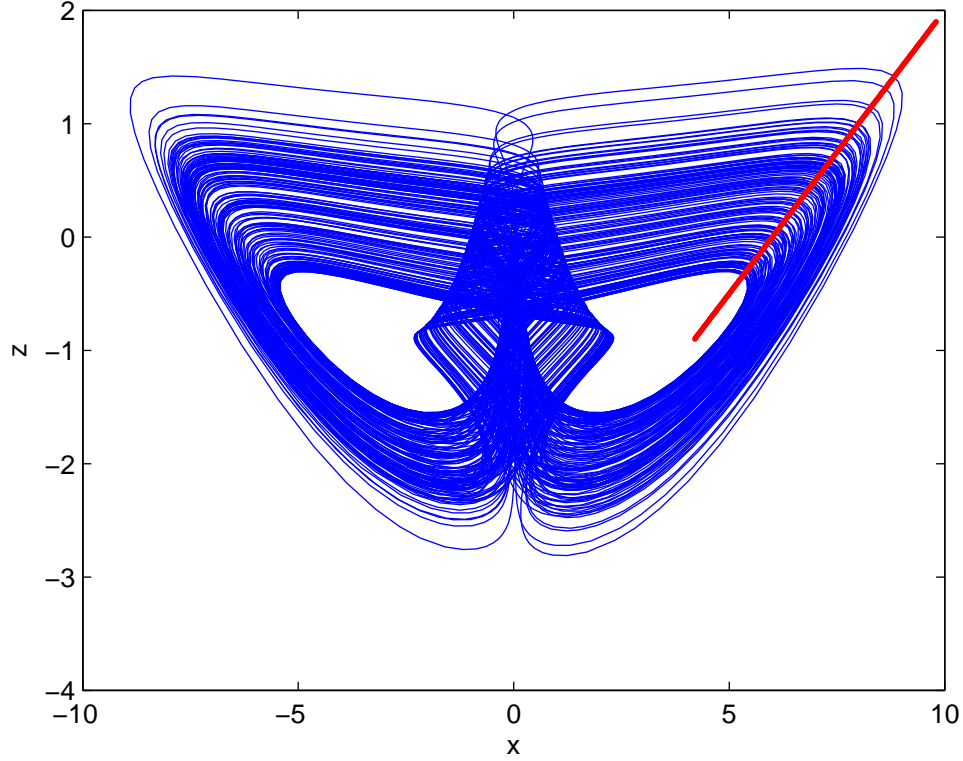
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# 1 Main section

In this note, we show that the main result proposed in [2], i.e. a sufficient condition for excluding the presence of homoclinic and heteroclinic orbits, is not correct. Hence this cannot lead to their conjecture of a fourth kind of chaos in 3D polynomial ODE systems characterized by the non-existence of homoclinic and heteroclinic orbits. Moreover, we remark that the conjecture can not be correct, as explained below in detail.

The main result of [2] is stated in their Theorem 1, which leads to exclude the existence of a bounded trajectory for  $t < t_0$  in any dynamical system characterized by a vector field with at least one lower bounded component, and from their proof it follows that this occurs independently of the existence of one or more equilibria in the system. We notice that the proof they give also implies the *non existence of any closed orbit, i.e. limit cycle*. Moreover, [2] gives an example, in eq. (3), satisfying the assumption of the Theorem 1 and showing a chaotic attractor illustrated in Fig. 1. From this finding, they conjecture the existence of a new type of chaos. However, given a system with a chaotic attractor (and let us assume that this is the case shown in their example, indeed from Fig. 1 of this note it can be assumed that the first return map on a suitable two-dimensional surface as qualitatively shown by the red line, leads to a two-dimensional map in chaotic regime) then we have, by any definition of chaotic system (see, e.g., [5], [6], [3], [4] and [1]), *the existence of infinitely many unstable limit cycles which densely cover the observed chaotic set. Moreover, infinitely many homoclinic and heteroclinic orbits exist connecting these unstable limit cycles, which also are dense in the chaotic attractor*. It follows that it is not possible to identify a chaotic system characterized by the non-existence of homoclinic and

heteroclinic orbits.



**Figure 1:** Projection onto  $xz$ -plane of attractor obtained from system (3) in [2] for  $a = 40$ ,  $b = 32$  and  $c = 10$ . The red line indicates a suitable plane for the return map.

The paper [2] suggests that the presence of chaos is subordinated to the existence of either homoclinic orbits *of equilibria* or heteroclinic orbits *of equilibria*. Which is not correct. Indeed, the target of identifying a chaotic system characterized by the non-existence of homoclinic and heteroclinic orbits *to one or more equilibria* may be correct. However, in our opinion this is not interesting, as it is well known that chaotic attractors may exist also when neither homoclinic nor heteroclinic orbits *to equilibria* are present,

as it is the case in the classical Lorenz system, see, e.g. [7]. We should emphasize that the existence of a homoclinic orbit of an equilibrium (or heteroclinic connections between two equilibria), under other suitable assumptions, is relevant from a theoretical point of view, as it allows to rigorously prove the existence of chaos and also its persistence under perturbations when the homoclinic (heteroclinic) orbit no longer exists.

In the following, we underline the presence of an inaccuracy in the proof of Theorem 1 in [2].

**Remarks on the proof of Theorem 1 in [2].** Regarding the proof of Theorem 1 in [2], an incorrect part is related to the unboundedness of the trajectories. Consider a vector field  $f = (f_1, f_2, \dots, f_n)^T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  belonging to class  $C^r$  ( $r \geq 1$ ),  $x = (x_1, x_2, \dots, x_n)^T$  the state variable of the system, and  $t \in \mathbb{R}$  the time. Assuming the existence of an  $\alpha < 0$ , such that for at least one  $j \in \{1, 2, \dots, n\}$  we have that  $f_j(x) \geq \alpha$ ,  $\forall x \in \mathbb{R}^n$ , then (as pointed out by the authors in eq. (2) of their paper) we have to consider the inequality

$$x_j(t) \geq \alpha(t - t_0) + x_j(t_0) \quad (1)$$

**for  $t \geq t_0$** , which implies that, given a homoclinic or heteroclinic orbit  $(\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t))$ , we have  $\lim_{t \rightarrow +\infty} \gamma_j(t) \geq -\infty + \gamma_j(t_0)$ , which is compatible with the existence of the orbits itself. Then the authors state that the same result in (1) holds also **for  $t < t_0$** , getting divergence for any orbit for  $t \rightarrow -\infty$ . However this is not correct, as for  $t < t_0$  we have<sup>1</sup>

$$x_j(t) \leq \alpha(t - t_0) + x_j(t_0) \quad (2)$$

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<sup>1</sup>Note that, given

$$\dot{x}(t)_j = f_j(x(t)),$$

in place of (1). Inequality (2) implies that  $\lim_{t \rightarrow -\infty} \gamma_j(t) \leq +\infty + \gamma_j(t_0)$ , which does not exclude the existence of homoclinic (or heteroclinic) orbits. ■

## 2 Conclusions

In this note we provide some arguments showing that the statement of Theorem 1 in [2] is not correct. Moreover, we show the presence of a mistake related to the backward integration in the proof of the same Theorem.

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by a simple integration from  $t_0$  to  $t$ , we have

$$x_j(t) = x_j(t_0) + \int_{t_0}^t f_j(x(s)) ds$$

Assuming  $f_j(x) \geq \alpha \forall x \in \mathbb{R}$ , by basic properties of definite integrals we have that for  $t > t_0$

$$\int_{t_0}^t f_j(x(s)) ds \geq \alpha(t - t_0)$$

from which inequality (1) follows, and for  $t < t_0$

$$\int_{t_0}^t f_j(x(s)) ds = - \int_t^{t_0} f_j(x(s)) ds \leq -\alpha(t_0 - t) = \alpha(t - t_0)$$

from which we obtain inequality (2).

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